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Alias-Free Sampling of Random Noise

HAROLD S. SHAPIRO and RICHARD A. SILVERMAN

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Harold S. Shapiro and Richard A. Silverman

H S Shapiro
Harold S. Shapiro

Richard A. Silverman
Richard A. Silverman

Sidney Borowitz
Sidney Borowitz
Acting Director

Dr. Werner W. Gerbes
Dr. Werner Gerbes
Contract Monitor

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Abstract

We study the question of when the power spectrum of a stationary random process is uniquely determined by the values of the process measured in various ways at discrete instants of time. As is well known, in the case of equi-spaced sampling, the power spectrum is not uniquely determined by the sampled data, i.e. aliasing is present. After a review of the equi-spaced case, we examine the case where the time markers are not equi-spaced, but are subject to jitter, and find that in general aliasing persists. We then consider the case where the time markers are generated by an additive scheme, whereby each sampling time is derived from the previous one by the addition of an independent random variable, with characteristic function $\phi(s)$. It is found that additive random sampling is alias-free if $\phi(s)$ is one-to-one on the real axis, but not alias-free if $\phi(s)$ takes the same value at two distinct points of the open upper half-plane. Various alias-free sampling methods are exhibited, notably Poisson sampling.

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1. Introduction.

A problem of great practical interest is the determination of the power spectrum $F(\omega)$ of a stationary random process $x(t)$ from measurements of $x(t)$ made at discrete instants of time. As is well known (see e.g. [1]) the usual method of sampling at equally spaced instants of time $t_n = nh$, $n = \ldots, -2, -1, 0, 1, 2, \ldots$, does not permit unambiguous determination of $F(\omega)$, unless $F(\omega)$ is known in advance to lie in the Nyquist band N, defined by $|\omega| \leq \pi/h$; otherwise, a whole class of different power spectra are compatible with the sampled values of $x(t)$. In the usual terminology, these different power spectra are called aliases of $F(\omega)$*. Thus, it is clear that if we hope to find alias-free sampling methods, i.e. methods which lead to unambiguous determination of $F(\omega)$, we must sample at unequally spaced instants of time. The case where these unequally spaced times are generated purely arithmetically seems to lead to great analytical difficulties**. However, as shown in the present paper, the case where the sampling times are chosen randomly is quite tractable. In fact, we shall see that some random sampling schemes succeed in eliminating aliasing, while others do not. In particular, the important case of Poisson sampling, where the sampling times are taken to be the occurrence times of the events of some Poisson process (e.g. the arrival times of shots in shot noise or the particle emission times in a radioactive decay process) is found to be alias-free.

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*Equivalently, we call the corresponding correlation functions aliases of one another.

**See remark on p. 219 of [1].

2. Periodic Sampling.

In what follows we shall be concerned exclusively with second order (or wide sense) properties of $x(t)$, which may therefore be taken to be a stationary Gaussian process. We assume that $x(t)$ is real, with mean zero and continuous correlation function $C(\tau)$, i.e.

$$Ex(t) = 0 \quad , \quad Ex(t)x(t') = C(t-t') \quad ,$$

where E denotes the expectation or ensemble average*. By the usual argument, $C(-\tau) = C(\tau)$ and $|C(\tau)| \leq C(0)$. We write the Wiener-Khintchine relations in the form

(1) $$C(\tau) = \int_{-\infty}^{\infty} \exp(i\omega\tau)F(\omega)d\omega \quad , \quad F(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \exp(-i\omega\tau)C(\tau)d\tau \, ,$$

where the even, non-negative function $F(\omega)$ is the power spectrum (more accurately, the power spectral density) of $x(t)$. It will be assumed that $F(\omega)$ is not only integrable, as follows from $\int_{-\infty}^{\infty} F(\omega)d\omega = Ex^2(t)$,

but also square-integrable. Symbolically**, we have $F(\omega) \in L^1 \cap L^2$. It follows by Plancherel's theorem that $C(\tau) \in L^2$, and that the second integral in (1) may have to be interpreted as $\underset{R\to\infty}{\text{l.i.m.}} \int_{-R}^{R}$.

---

*We shall always assume that $x(t)$ has the ergodicity needed to replace the operation E by suitable infinite time averages. In the case of Gaussian $x(t)$, it follows by a theorem of Maruyama [2, Thm. 4] that $x(t)$ is ergodic, since the required continuity of the spectral distribution function of $x(t)$ is implicit in the representation (1). In this first study of alias-free sampling, we shall not study the variance of finite-time power spectrum estimates (as done e.g. in [1]).

**By $f(\omega) \in L^p(a,b)$, $p = 1,2$, we mean as usual that the (Lebesgue) integral $\int_a^b |f(\omega)|^p d\omega$ is finite. We abbreviate $L^p(-\infty,\infty)$ to L^p.

Suppose now that we sample $x(t)$ periodically (i.e. with equal spacing) at the points $t_n = nh$, $n = \ldots, -2, -1, 0, 1, 2, \ldots$, where $h > 0$ is the sampling interval. This generates the stationary random sequence $x(t_n) = x(nh)$, with mean zero and correlation function (equivalently, correlation sequence)

$$c_h(n) = C(nh) \quad , \quad n = \ldots, -2, -1, 0, 1, 2, \ldots$$

Since $C(\tau)$ is even, we have $c_h(-n) = c_h(n)$. The discrete analogs of (1), which will be needed later, are

$$(2) \qquad c_h(n) = \int_{-\pi}^{\pi} \exp(i\omega n) f_h(\omega) d\omega \quad , \quad f_h(\omega) = \frac{1}{2\pi} \sum_{n=-\infty}^{\infty} \exp(-i\omega n) c_h(n) \quad ,$$

where $f_h(\omega)$, the power spectrum of $x(t_n)$, is defined only in the interval $(-\pi, \pi)$.

We now give two different ways of constructing aliases for the case of periodic sampling. The first method relies on the fact that any two correlation functions which agree on the "lattice" of points $t_n = nh$ are obviously aliases with respect to periodic sampling with spacing h. To construct such a pair of correlation functions, let $h(\tau)$ denote any real even function in $L^1 \cap L^2$, possessing a continuous second derivative and vanishing at the points $t_n = nh$*. Then the

---

*We exclude the trivial case $h(\tau) \equiv 0$.

Fourier transform

$$H(\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \exp(-i\omega\tau)h(\tau)d\tau$$

is real, even and in $L^1 \cap L^2$ (since $H(\omega) = O(|\omega|^{-2})$ for large ω). If now we write

$$(3) \qquad \begin{aligned} H_+(\omega) &= H(\omega) \quad , \quad H(\omega) \geq 0 \quad , \\ H_+(\omega) &= 0 \qquad\quad , \quad H(\omega) < 0 \quad , \end{aligned}$$

and

$$(4) \qquad \begin{aligned} H_-(\omega) &= -H(\omega) \ , \quad H(\omega) < 0 \quad , \\ H_-(\omega) &= 0 \qquad\quad , \quad H(\omega) \geq 0 \quad , \end{aligned}$$

then $H_+(\omega)$ and $H_-(\omega)$ are a pair of even, non-negative functions in $L^1 \cap L^2$, whose Fourier transforms $C_+(\tau)$ and $C_-(\tau)$ agree on the lattice $t_n = nh$, and hence are aliases of each other.* Note that $H_+(\omega)$ and $H_-(\omega)$ are not only distinct, but even non-overlapping. However, this method can also be used to construct aliases which lie in the same frequency band. We merely note that all the functions

$$(5) \qquad H_a(\omega) = aH_+(\omega) + (1-a)H_-(\omega) \quad , \quad 0 \leq a \leq 1 \quad ,$$

are aliases of one another. In fact, more generally, any element of

*By construction, $C_+(\tau)$ and $C_-(\tau)$ are automatically positive definite, and hence correlation functions.

the convex hull* of a set of alias power spectra is another alias.

Using the same construction, we can show that knowing the correlation functions of the finite family of random sequences $x(nh_1),\ldots,x(nh_m)$ is not enough to eliminate aliases, i.e., there exist pairs of correlation functions which agree on any finite number of lattices of equally spaced points. It is merely necessary to choose the function $h(\tau)$ in the above construction to be any real even function in $L^1 \cap L^2$, possessing a continuous second derivative and vanishing on all the lattices $nh_1,\ldots,nh_m$, and then construct $H_+(\omega)$ and $H_-(\omega)$ as before. Again, more aliases, with overlapping spectra, can be constructed by using (5).

The second method of constructing aliases is based on a detailed examination of the relation between the power spectrum $f_h(\omega)$ of the random sequence $x(nh)$ and the power spectrum $F(\omega)$ of the underlying process $x(t)$. Unlike the first method, it can be used to construct aliases for any given power spectrum (or, equivalently, for any given correlation function). We begin by writing (see [3], p. 57)

$$
\begin{aligned}
c_h(n) = C(nh) &= \sum_{k=-\infty}^{\infty} \int_{2\pi k\rho}^{2\pi(k+1)\rho} \exp(i\omega nh)F(\omega)\,d\omega \\
&= \int_{-\pi\rho}^{\pi\rho} \exp(i\omega nh)\left\{\sum_{k=-\infty}^{\infty} F(\omega+2\pi k\rho)\right\} d\omega \quad ,
\end{aligned}
\tag{6}
$$

*By the convex hull of a set of functions f_n, $1 \leq n \leq N$, is meant the set of all functions of the form $\sum_{n=1}^{N} a_n f_n$, $0 \leq a_n \leq 1$, where $\sum_{n=1}^{N} a_n = 1$.

where

$$\rho = 1/h$$

is the sampling rate. At this point, it is convenient to introduce the linear operator $\mathcal{W} \equiv \mathcal{W}_\rho$ *, defined by

$$(7) \qquad \mathcal{W} f(\omega) = \sum_{k=-\infty}^{\infty} f(\omega + 2\pi k\rho) \quad .$$

The following properties of the operator $\mathcal{W}$ are easily verified:

1. If $f(\omega) \in L^1$, then $\mathcal{W} f(\omega)$ exists for almost all ω and has period $2\pi\rho$**.

2. If $f(\omega) \in L^1$, then $\mathcal{W} f(\omega) \in L^1(-\pi\rho, \pi\rho)$, and in fact

$$\int_{-\pi\rho}^{\pi\rho} \mathcal{W} f(\omega) d\omega = \int_{-\infty}^{\infty} f(\omega) d\omega \quad .$$

3. If $f(\omega)$ is continuous and $f(\omega) = O(|\omega|^{-1-\varepsilon})$, $\varepsilon > 0$, then $\mathcal{W} f(\omega)$ is continuous**.

4. $\mathcal{W} f(\omega + 2\pi n\rho) = \mathcal{W} f(\omega)$, where n is any integer.

5. If $h(\omega)$ has period $2\pi\rho$, then $\mathcal{W} h(\omega) f(\omega) = h(\omega) \mathcal{W} f(\omega)$.

Suppose now that $f(\omega) \geq 0$, as in the case of power spectra. Then $f(\omega)$ can be regarded as a distribution of mass on the ω-axis, and $\mathcal{W} f(\omega)$ is the mass distribution on the circumference of a circle of radius ρ

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*The letter $\mathcal{W}$ is meant to suggest "winding" (see below).

**In this connection, note that Wiener's class M_1, defined in [4], p. 73, is essentially the class of continuous L^1 functions whose "wound-up" versions are finite (and hence continuous).

which results when the ω-axis is wound up on the circle in such a way that the point $\omega = 0$ on the line coincides with the point $\omega = 0$ on the circle; in this construction, superimposed mass simply adds. Comparing (2), (6) and (7), we see at once that the relation between the power spectrum $f_h(\omega)$ of the random sequence $x(nh)$ and the power spectrum $F(\omega)$ of the underlying process $x(t)$ is just $f_h(\omega) = \rho\mathcal{W}F(\rho\omega)$, i.e., $f_h(\omega)$ is the "wound-up" version of $F(\omega)$, measured in angular units. Thus, any other power spectrum $\hat{F}(\omega)$ which gives the same mass distribution when wound up on a circle of radius ρ is an alias of $F(\omega)$, with respect to periodic sampling at rate ρ, and conversely, all aliases of $F(\omega)$ wind up to the same mass distribution. From this point of view, we can create any number of aliases of $F(\omega)$ by shifting some or all of the spectrum $F(\omega)$ through integral multiples of $2\pi\rho$. More precisely, given any decomposition $F(\omega) = \sum_{n=1}^{\infty} F_n(\omega)$, where the $F_n(\omega)$ are all non-negative, construct the new spectrum

$$\hat{F}(\omega) = \frac{1}{2}\sum_{n=1}^{\infty}\left\{F_n(\omega + 2\pi k_n\rho) + F_n(-\omega + 2\pi k_n\rho)\right\},$$

where the k_n are arbitrary integers. Then $\hat{F}(\omega)$ is even, non-negative and integrable, and moreover

$$\mathcal{W}\hat{F}(\omega) = \frac{1}{2}\sum_{n=1}^{\infty}\left\{\mathcal{W}F_n(\omega + 2\pi k_n\rho) + \mathcal{W}F_n(-\omega + 2\pi k_n\rho)\right\}$$

$$= \frac{1}{2}\sum_{n=1}^{\infty}\left\{\mathcal{W}F_n(\omega) + \mathcal{W}F_n(-\omega)\right\} = \frac{1}{2}\left\{\mathcal{W}F(\omega) + \mathcal{W}F(-\omega)\right\} = \mathcal{W}F(\omega),$$

as required.

We note in passing that the technique developed above can also be used to construct correlation functions which vanish at all points of the sequence $t_n = nh$, $n \neq 0$. For suppose $F(\omega)$ is any power spectrum for which $\mathcal{M}F(\omega)$ has a positive lower bound*. Then the power spectrum $H(\omega) = F(\omega) / \mathcal{M}F(\omega)$ obviously satisfies $\mathcal{M}H(\omega) \equiv 1$, whence it follows from (2) that

$$c_h(n) = C(nh) = 2\pi\rho\delta_{on} \quad ,$$

where δ_{mn} is the Kronecker delta, i.e. $C(nh) = 0$ for $n \neq 0$, as required.

3. <u>Jittered periodic sampling.</u> When trying to implement the periodic sampling scheme, we have to expect deviations of the sampling times from the values nh, due to jitter in the device generating the time markers. We now inquire as to the effect that such inaccuracy has on aliasing; in particular, it is conceivable that the jitter actually present, or some extra jitter deliberately introduced, might eliminate the aliasing encountered with ideal periodic sampling. This leads us to consider the following modification of ideal periodic sampling: Let the sampling times be of the form $t_n = nh + \gamma_n$, $n = \ldots, -2, -1, 0, 1, 2, \ldots$, where the γ_n are a family of independent, identically distributed Gaussian random variables, with mean zero and variance σ^2, i.e. with

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*In particular, this implies that $\int_{-\pi}^{\pi} \log f_h(\omega)\, d\omega > -\infty$, so that the random sequence $x(t_n)$ is <u>indeterministic</u> (see e.g. [3], p. 69).

common probability density

$$p(\tau) = \frac{1}{\sqrt{2\pi}\,\sigma} \exp(-\tau^2/2\sigma^2).$$

The condition for small jitter is $\sigma \ll h$; in the limit $\sigma = 0$, $p(\tau)$ reduces to $\delta(\tau)$, the Dirac delta function, and we have ideal periodic sampling. Note that $Et_n = nh$, so that the average spacing between sampling times is still h.

Since there is no statistical dependence between the γ_n and $x(t)$, the zero-mean random sequence $x(t_n)$ is stationary (with respect to the product statistics of the γ_n and $x(t)$) with correlation function*

$$\begin{aligned} c_h(n) &= Ex(t_{m+n})x(t_m) = E\left[Ex(t_{m+n})x(t_m); t_{m+n} \text{ and } t_n \text{ fixed}\right] \\ &= \int_{-\infty}^{\infty} C(\tau)\, p_n(\tau)\, d\tau \quad , \end{aligned} \tag{8}$$

where $p_n(\tau)$, the probability density of $t_{m+n} - t_m$, is independent of m. For $n = 0$, we have $p_o(\tau) = \delta(\tau)$, and (8) reduces to $c_h(0) = C(0)$, a relation which is obvious from the fact that the mean square of any sequence sampled from $x(t)$ must be the same as the mean square of $x(t)$ itself.

*See [3], p. 58. The slight abuse of notation in (8) is transparent, i.e., the first E is an average over the product statistics of the γ_n and $x(t)$, whereas the second E is an average over the statistics of the γ_n and the third E over the statistics of $x(t)$.

Now if $g(\tau)$ is the probability density of a Gaussian random variable with mean zero and variance $2\sigma^2$, i.e.

(9) $$g(\tau) = \frac{1}{2\sqrt{\pi}\,\sigma} \exp\left(-\tau^2/4\sigma^2\right) \quad ,$$

we see at once that

(10) $$c_h(n) = \int_{-\infty}^{\infty} g(nh-\tau)\, C(\tau)d\tau \quad , \quad n \neq 0$$

Since $g(nh-\tau)$ does not reduce to $\delta(\tau)$ for $n = 0$, (10) is not valid for $n = 0$; instead, as already noted, we have $c_h(0) = C(0)$. However, we shall assume that the quantity

(11) $$d_h(0) = \int_{-\infty}^{\infty} g(\tau)C(\tau)d\tau \quad ,$$

obtained by setting $n = 0$ in the right-hand side of (10), is known from a suitable side experiment. For example, we might sample not only at the points t_n, but also at the points $t_n + h$, and calculate*

(12) $$d_h(0) = \lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} x(t_n + h)x(t_{n+1}) \quad .$$

Or we might mark off two sets of random times $t_n = nh + \gamma_n$, $t_n' = nh + \gamma_n'$, where the γ_n' are another set of independent Gaussian

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*There is perhaps a slight contradiction in (12), for although the times t_n are subject to jitter, it is assumed that no additional jitter occurs in marking off the times $t_n + h$.

random variables with common distribution (9), and calculate

$$d_h(0) = \lim_{N \to \infty} \frac{1}{N} \sum_{n=1}^{N} x(t_n)x(t_n') \quad .$$

(The $c_h(n)$, $n \neq 0$, can be calculated from either the t_n or the t_n'.) Our primary justification for assuming that $d_h(0)$ is known is that it is easily calculable from an <u>ensemble</u> point of view. In fact, indicating a "typical" infinite set of sample functions of the process $x(t)$ by $x_i(t)$, $1 \leq i < \infty$, we have

$$d_h(0) = \lim_{N \to \infty} \frac{1}{N} \sum_{n=1}^{N} x_i(t_n)x_{i+1}(t_n) \quad .$$

The issue involved is that the various points labelled t_n (n fixed) are scattered in time when looked at across the ensemble. In any event, since, as we shall see below, aliasing is in general present even when $d_h(0)$ is known, it is present <u>a fortiori</u> in the absence of this extra information.

We observe next that the convolution of $C(\tau)$ with $g(\tau)$, i.e.

$$D(\tau) = \int_{-\infty}^{\infty} g(\tau - u)C(u)du \tag{13}$$

is itself a correlation function, as follows at once from the fact that it has the non-negative Fourier transform $H(\omega) = F(\omega)G(\omega)$, where

$$G(\omega) = \exp(-\sigma^2\omega^2/2) \quad .$$

Thus the sequence

$$d_h(n) = c_h(n) \quad , \qquad n \neq 0 \quad ,$$

$$d_h(0) = \int_{-\infty}^{\infty} g(\tau)C(\tau)d\tau \quad ,$$

is clearly a correlation sequence. In fact, if $y(t)$ denotes a zero-mean random process with correlation function $D(\tau)$, then $d_h(n)$ is the correlation function of the random sequence $y(nh)$ obtained by periodic sampling of $y(t)$ with spacing h. Incidentally, this gives an independent proof that the sequence $c_h(n)$ is a correlation sequence, since

$$c_h(n) = d_h(n) + a\delta_{on} \quad ,$$

where

$$a = \int_{-\infty}^{\infty} F(\omega) \left\{1 - G(\omega)\right\} d\omega > 0 \quad .$$

Thus, $c_h(n)$ is the sum of two correlation functions and therefore itself a correlation function.

The question of whether or not aliasing occurs with jittered periodic sampling can now be stated simply as follows: Are there two (or more) correlation functions which take the same value at $\tau = 0$ and which when convoled with $g(\tau)$ give the same values on the lattice $\tau = nh$, $n = \ldots, -2,-1,0,1,2,\ldots$? Or, in spectral terms, are there two (or more) power spectra which have the same total mass and which

when multiplied by $G(\omega)$ give the same "wound-up" version on a circle of radius ρ? (Note that both statements reduce to the case of periodic sampling when $\sigma = 0$.) As we shall now show, the answer is that aliasing persists in the presence of jitter (with an exception to be discussed). We now proceed to give two ways of constructing aliases, which parallel the methods used before for constructing aliases in the case of ideal periodic sampling.

The first method consists of finding two correlation functions $D_+(\tau)$ and $D_-(\tau)$* which vanish on the lattice $\tau = nh$, $n = \ldots,-2,-1,0,1,2,\ldots$, and whose Fourier transforms have the same integral when divided by $G(\omega)$. Since $G(\omega) \to 0$ as $\omega \to \pm\infty$, we shall arrange for $H_+(\omega)$ and $H_-(\omega)$, the Fourier transforms of $D_+(\tau)$ and $D_-(\tau)$, to vanish outside of some finite interval. This can be achieved by choosing $D_+(\tau)$ and $D_-(\tau)$ to be entire transcendental functions of exponential type, whence it follows by a theorem of Paley and Wiener (see e.g. [5] p. 134) that $H_+(\omega)$ and $H_-(\omega)$ vanish outside of some finite interval.

With this in mind, define the function

$$h_\alpha(\tau) = \left(\frac{\sin \pi\alpha\tau}{\tau}\right)^2 \sin^2 \pi\rho\tau \quad ,$$

where α is an arbitrary real number and $\rho = 1/h$ is the sampling rate. Then form the linear combination

$$h(\tau) = A\, h_\alpha(\tau) + h_\beta(\tau) \quad ,$$

*The notation anticipates the nature of the construction.

where $\alpha \neq \beta$ and A is a constant to be chosen later. The function $h(\tau)$ is real, even and in $L^1 \cap L^2$, and vanishes at all points of the lattice $\tau = nh$. Its Fourier transform $H(\omega)$ is therefore real, even and in L^2. Moreover, $h(\tau)$ is an entire function of exponential type with exponent $\omega_o = 2\pi(\rho + \max(\alpha,\beta))$. Hence, by the Paley-Wiener theorem, $H(\omega)$ vanishes for $|\omega| > \omega_o$, so that both $H(\omega)$ and $H(\omega)/G(\omega)$ are in L^1. We now choose the constant A so that

$$\int_{-\infty}^{\infty} \frac{H(\omega)}{G(\omega)}\, d\omega = 0 \quad .$$

Since the functions $h_\alpha(\tau)$ and $h_\beta(\tau)$ are linearly independent, the resulting $h(\tau)$ cannot vanish identically. Now define the non-negative functions $H_+(\omega)$ and $H_-(\omega)$ exactly as in equations (3) and (4) of the preceding section, and let their Fourier transforms be $D_+(\tau)$ and $D_-(\tau)$, which are automatically correlation functions. Then $D_+(\tau)$ and $D_-(\tau)$ agree on the lattice $\tau = nh$, and moreover, by construction the corresponding correlation functions $C_+(\tau)$ and $C_-(\tau)$, defined by (13), agree for $\tau = 0$. This completes the construction of aliases by the first method.

The second method of constructing aliases is based as before on the winding construction, and has the advantage that it allows one to produce aliases for any given power spectrum. Using the correlation sequence $d_h(n)$ we form the function

$$\hat{H}(\omega) = \frac{1}{2\pi\rho} \sum_{n=-\infty}^{\infty} \exp(-i\omega nh) d_h(n) = F(\omega)G(\omega) \quad , \quad \omega \in N ,$$

$$\hat{H}(\omega) = 0 \;, \quad \omega \notin N,$$

where N is the Nyquist band, defined by $|\omega| \leq \pi\rho$. We also have available the quantity $c_h(0) = \int_{-\infty}^{\infty} F(\omega)d\omega$. Now evaluate the integral

$$I = \int_{-\pi\rho}^{\pi\rho} \frac{\hat{H}(\omega)}{G(\omega)}\, d\omega \quad .$$

We have

$$I = \int_{-\pi\rho}^{\pi\rho} \frac{1}{G(\omega)}\left\{\sum_{n=-\infty}^{\infty} F(\omega + 2\pi\rho n)G(\omega + 2\pi\rho n)\right\} d\omega$$

$$\leq \int_{-\pi\rho}^{\pi\rho}\left\{\sum_{n=-\infty}^{\infty} F(\omega + 2\pi\rho n)\right\} d\omega = \int_{-\infty}^{\infty} F(\omega)d\omega = c_h(0) \quad ,$$

where the equality holds if and only if $F(\omega)$ lies entirely in N (i.e. vanishes almost everywhere outside of N). Thus, if the values of I and $c_h(0)$ are equal, we know that $F(\omega)$ lies in N; consequently, $F(\omega) = \hat{H}(\omega)/G(\omega)$ and we can determine $F(\omega)$ uniquely in this case. Moreover, unlike the case of ideal (i.e. jitter-free) periodic sampling, we can ascertain whether or not $F(\omega)$ lies in N by a simple test, namely comparison of I and $c_h(0)$, instead of having to know a priori that $F(\omega)$ lies in N.

If $F(\omega)$ does not lie in N, then $I < c_h(0)$, and aliases of $F(\omega)$ can be constructed as follows. Define the "virtual spectrum"

$$\hat{F}(\omega) = \hat{H}(\omega)/G(\omega) \quad , \qquad \omega \in N \quad ,$$

$$\hat{F}(\omega) = 0 \quad , \qquad \omega \notin N \quad ;$$

$\hat{F}(\omega) = F(\omega)$ if and only if $F(\omega)$ lies in N. Now write $\hat{F}(\omega)$ in any way* as the sum of two non-negative functions $\hat{F}_1(\omega)$ and $\hat{F}_2(\omega)$, and define

$$F_{n,\alpha}(\omega) = \frac{1}{2}\left\{\hat{F}_1(\omega) + \hat{F}_1(-\omega)\right\} + \frac{1-\alpha}{2}\left\{\hat{F}_2(\omega) + \hat{F}_2(-\omega)\right\}$$

$$+ \frac{\alpha}{2G(\omega)}\left\{\hat{F}_2(\omega + 2\pi\rho n)G(\omega + 2\pi\rho n) + \hat{F}_2(-\omega + 2\pi\rho n)G(-\omega + 2\pi\rho n)\right\} ,$$

where $0 \leq \alpha \leq 1$ and n is arbitrary. (Note that $F_{n,o}(\omega) = \hat{F}(\omega)$.) Then $F_{n,\alpha}(\omega)$ is even and non-negative, and moreover

$$\mathcal{W} F_{n,\alpha}(\omega)G(\omega) = \mathcal{W} F(\omega)G(\omega) = \hat{H}(\omega) \quad .$$

Since $F_2(\omega)$ vanishes for $\omega \notin N$, it is easily seen that if $n \neq 0$

$$\int_{-\infty}^{\infty} \frac{1}{G(\omega)} \left\{\hat{F}_2(\omega + 2\pi\rho n)G(\omega + 2\pi\rho n) + \hat{F}_2(-\omega + 2\pi\rho n)G(-\omega + 2\pi\rho n)\right\} d\omega$$

$$> \int_{-\infty}^{\infty}\left\{\hat{F}_2(\omega) + \hat{F}_2(-\omega)\right\} d\omega \quad ,$$

i.e., the distribution $F_{n,\alpha}(\omega)$ has more mass than the initial distribution $\hat{F}(\omega)$ if $0 < \alpha \leq 1$. In fact, since the first integral increases with n, there exists an integer $n_o > 0$ such that

$$\int_{-\infty}^{\infty} F_{n,1}(\omega)\, d\omega > \int_{-\infty}^{\infty} F(\omega) d\omega$$

---

*We exclude the trivial case $\hat{F}_1(\omega) \equiv 0$ or $\hat{F}_2(\omega) \equiv 0$.

for all $|n| \geq n_o$. Therefore, since

$$\int_{-\infty}^{\infty} F_{n,o}(\omega)d\omega = \int_{-\infty}^{\infty} F(\omega)d\omega < \int_{-\infty}^{\infty} F(\omega)d\omega \quad ,$$

and since $F_{n,\alpha}(\omega)$ depends continuously on the parameter α, there is a value of α, say α_n, between 0 and 1, such that

$$\int_{-\infty}^{\infty} F_{n,\alpha_n}(\omega)d\omega = \int_{-\infty}^{\infty} F(\omega)d\omega \quad .$$

It is clear that any such function $F_{n,\alpha_n}(\omega)$ is an alias of $F(\omega)$ with respect to jittered periodic sampling. This completes the construction of aliases by the second method.*

4. Additive random sampling.

The reason that aliasing is still present with jittered periodic sampling seems to be that the sampling times t_n are still "attracted" to the equi-spaced values nh. What seems to be needed to break up this regularity is some sort of "floating point" sampling scheme. Thus, we consider next additive random sampling, in which each sampling time is derived from the preceding one by the addition of an independent random variable**. Specifically, let $t_n = t_{n-1} + \gamma_n$, where the γ_n, $n = \ldots,-2,-1,0,1,2,\ldots$, are a family of identically distributed,

*The considerations of this section can be extended to the case where $g(\tau)$ instead of being Gaussian is a function whose Fourier transform has isolated zeros only, e.g. when $g(\tau)$ vanishes outside of a finite interval. Again it is found that aliasing persists. It is also clear how to extend the second method of constructing aliases to give aliases of a more general type.

**With jittered periodic sampling, the differences between successive sampling times are not independent random variables, and the variance of t_n is $2\sigma^2$ for all n. On the other hand, with additive random sampling, the variance of t_n goes to ∞ with n, if sampling starts in the finite past (whence the phrase "floating point"), although Et_n still equals nh.

independent random variables, with $E\gamma_n = h < \infty$ and common probability density $p(\tau)$. Of course, $p(\tau) \geq 0$ and $\int_{-\infty}^{\infty} p(\tau)d\tau = 1$. We assume that $p(\tau)$ is also in L^2 and that $p(\tau) = 0$ for $\tau < 0$. The last condition corresponds to the reasonable requirement that a sample time with a given index should come <u>after</u> all sample times with smaller indices. For later use, we introduce the Fourier transform

$$(14) \qquad \phi(\omega) = \int_{-\infty}^{\infty} \exp(i\omega\tau)p(\tau)d\tau = \int_{0}^{\infty} \exp(i\omega\tau)p(\tau)d\tau \quad ,$$

the characteristic function (in the sense of probability theory) of the distribution $p(\tau)$. The function $\phi(\omega)$ is uniformly continuous on the whole real line and satisfies the conditions

$$\phi(0) = 1 \ , \quad |\phi(\omega)| \leq 1 \ , \ \phi(-\omega) = \overline{\phi(\omega)}, \ \lim_{\omega \to \pm\infty} \phi(\omega) = 0 \quad ,$$

where the overbar denotes the complex conjugate. It will be important to extend (14) to complex arguments by writing

$$(15) \qquad \phi(s) = \int_{0}^{\infty} \exp(is\tau)p(\tau)d\tau \quad ,$$

where $s = \omega + i\omega'$. Eq.(15) defines a function which is analytic and bounded in the open half-plane $\operatorname{Im} s > 0$, and continuous in the closed half-plane $\operatorname{Im} s \geq 0$. Moreover, for $\operatorname{Im} s \geq 0$, we have $|\phi(s)| \leq 1$, with equality holding only for $s = 0$.

As in the case of jittered random sampling, the correlation function

of the random sequence $x(t_n)$ is given by

$$(8)\qquad c_h(n) = Ex(t_{m+n})x(t_m) = E\left[Ex(t_{m+n})x(t_m);\ t_{m+n} \text{ and } t_n \text{ fixed}\right]$$

$$= \int_{-\infty}^{\infty} C(\tau)p_n(\tau)d\tau \quad ,$$

where now, in view of the new definition of t_n, the probability density $p_n(\tau)$ is given by

$$(16)\qquad p_n(\tau) = \int_{-\infty}^{\infty} p_{n-1}(\tau-u)p(u)du = \int_0^{\tau} p_{n-1}(\tau-u)p(u)du \quad , \quad n \geq 2 \ ,$$

$$p_1(\tau) \equiv p(\tau) \quad ,$$

i.e., $p_n(\tau)$, $n \geq 2$, is obtained by successively convolving $p(\tau)$ with itself n times. As before, $p_o(\tau) = \delta(\tau)$ and $c_h(-n) = c_h(n)$. Note that $p_n(\tau)$, $n \geq 1$, vanishes for negative τ, so that (8) becomes

$$(17)\qquad c_h(n) = \int_0^{\infty} C(\tau)p_n(\tau)d\tau \quad , \qquad n \geq 1 \ ,$$

$$c_h(0) = C(0) \quad .$$

Moreover, using Parseval's theorem, we have

$$\int_0^{\infty} p_n^2(\tau)d\tau = \int_{-\infty}^{\infty} |\phi(\omega)|^{2n}d\omega \leq \int_{-\infty}^{\infty} |\phi(\omega)|^2 d\omega = \int_0^{\infty} p^2(\tau)d\tau \quad ,$$

so that $p_n(\tau)$, $n \geq 1$, is in the class L^2. Again applying Parseval's theorem and the convolution theorem for Fourier transforms, we find

that the spectral equivalent of (17) is

$$(18) \qquad c_h(n) = \int_{-\infty}^{\infty} F(\omega)\phi^n(\omega)\, d\omega \;, \qquad n \geq 0 \quad .$$

The reality of $c_h(n)$, as defined by (18), follows from $F(-\omega) = F(\omega)$ and $\phi(-\omega) = \overline{\phi(\omega)}$.

Equations (17) and (18) exhibit a direct relation between the correlation sequence $c_h(n)$ and the correlation function $C(\tau)$ or power spectrum $F(\omega)$, and we shall use these formulas, particularly (18), to study aliasing in the case of additive random sampling. It is of some interest, however, to examine the relation between the spectrum $F(\omega)$ and the spectrum $f_h(\omega)$ of the random sequence $x(t_n)$. To do this, we use (2) and (18) to write *

$$f_h(\omega) = \frac{1}{2\pi}\sum_{n=-\infty}^{\infty} \exp(-i\omega n)\; c_h(n) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(u)\left\{1 + \sum_{n=1}^{\infty} \phi^n(u)(e^{i\omega n} + e^{-i\omega n})\right\}du,$$

where the second series is convergent for all $u \neq 0$. Carrying out the summation and using the fact that $F(\omega)$ is even, we obtain

$$(19) \qquad \begin{aligned} f_h(\omega) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} F(u)\; \mathrm{Re}\left\{\frac{1+e^{i\omega}\phi(u)}{1-e^{i\omega}\phi(u)}\right\} du \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty} F(u)\left\{\frac{1-|\phi(u)|^2}{|1-e^{i\omega}\phi(u)|^2}\right\} du \quad , \end{aligned}$$

so that $f_h(\omega)$ exists and is non-negative everywhere in $(-\pi,\pi)$, except perhaps at $\omega = 0$, where (19) becomes indeterminate. Incidentally, this

*See [3], p. 58.

gives an independent proof of the fact that $c_h(n)$ is a correlation sequence, i.e. that the sequence $x(t_n)$ is stationary. Finally, we have

$$\int_{-\pi}^{\pi} f_h(\omega)\, d\omega = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(u)\left\{\int_{-\pi}^{\pi} \mathrm{Re}\left\{\frac{1+e^{i\omega}\phi(u)}{1-e^{i\omega}\phi(u)}\right\} d\omega\right\} du$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} F(u)\left\{\mathrm{Re}\int_{|z|=1} \frac{1+z\phi(u)}{1-z\phi(u)}\,\frac{dz}{iz}\right\} du = \int_{-\infty}^{\infty} F(\omega)\,d\omega \quad ,$$

as is to be expected.

The question of whether aliasing occurs with additive random sampling can now be simply stated as follows: For which $p(\tau)$ is there only one correlation function which leads _via_ (18) to a given correlation sequence $c_h(n)$? Or in spectral terms, for which $p(\tau)$ is there only one power spectrum which leads _via_ (19) to $c_h(n)$? We now prove a theorem which gives a partial answer to this question, i.e. which gives a sufficient condition for additive random sampling to be alias-free.

Theorem 1. _Additive random sampling is alias-free if the characteristic function $\phi(\omega)$ takes no value more than once on the real axis._

Proof. If aliasing is possible, then (18) is satisfied for two distinct real (non-negative) functions in $L^1 \cap L^2$, i.e., there exists a real function $H(\omega)$ in $L^1 \cap L^2$, which is not a _null function_*, such that

$$(20) \qquad \int_{-\infty}^{\infty} H(\omega)\phi^n(\omega)\,d\omega = 0$$

*By a null function is meant a function which vanishes except on a set of measure zero.

for $n \geq 0$. Thus, we must show that if $\phi(\omega)$ is one-to-one on the real axis, then any $H(\omega)$ satisfying (20) must be a null function.

We begin by showing that if $\phi(\omega)$ is one-to-one on the real axis, then $\phi(\omega)$ is actually one-to-one for $\text{Im}s \geq 0$, or, in the terminology of conformal mapping, $\phi(s)$ is <u>schlicht</u> in the closed upper half plane. To see this, we note first that it follows by a Phragmén-Lindelöf theorem (see [6], p. 179) that the boundedness of $\phi(\omega)$ together with $\phi(\omega) \to 0$ as $\omega \to \pm\infty$ implies that $\phi(s) \to 0$ uniformly in the upper half-plane, i.e. that $\phi(s)$ can be made as small as we please outside a suitably large semicircle Γ_1 erected on some diameter $(-R,R)$. Let Γ be the closed contour consisting of Γ_1 and the segment $(-R,R)$. Now if $\alpha \neq 0$ is a value taken by $\phi(s)$ at some point in $\text{Im}s > 0$, choose R large enough so that $|\phi(s)| < \alpha$ on Γ_1. By the argument principle (see [6], p. 116), if $\phi(s)$ takes the value α more than once for $\text{Im}s > 0$, then the image of the contour Γ under the mapping $z = \phi(s)$ would have to wind more than once around the point α. However, this is impossible, since by hypothesis the image of the segment $(-R,R)$ of the real axis cannot intersect itself, whereas the image of Γ_1 is confined to a circle of radius less than α. Moreover, if $\phi(s)$ takes the value α at some point s_0 of $\text{Im}s > 0$, it cannot take the same value at a point ω_0 of the real axis, for since $\phi(s)$ takes all values sufficiently close to α in a neighborhood of s_0 (see [7], p. 184), and since $\phi(s)$ is continuous at ω_0, we could then find two non-zero values near α taken twice in the open upper half-plane, which is impossible, as we have just seen. It follows that $\phi(s)$ can take no <u>non-zero</u> value more than once in $\text{Im}s \geq 0$.

Moreover, $\phi(s)$ cannot vanish for $\operatorname{Im} s > 0$, for if $\phi(s_o) = 0$, $\operatorname{Im} s_o > 0$, then in a neighborhood of s_o which lies entirely in the open upper half-plane $\phi(s)$ takes all sufficiently small values. But since $\phi(s) \to 0$ as $s \to \infty$, we could then find a small non-zero value taken twice in the open upper half-plane, which is impossible, as already noted. Moreover, $\phi(\omega)$ cannot vanish for real ω, since then $\phi(-\omega)$ also vanishes, which violates the hypothesis of the theorem (recall that $\phi(0) \neq 0$.) Thus, we have finally shown that $\phi(s)$ is schlicht in the closed upper half plane.

We now turn to the core of the proof. Consider the analytic function $z = \phi(s)$, which maps the half-plane $\operatorname{Im} s > 0$ onto a domain Ω_z in the z-plane which is entirely contained in the circle $|z| = 1$ and whose boundary does not intersect itself and passes through the points $z = 0$ and $z = 1$. Next consider the analytic function

$$\zeta = \xi(s) = \frac{1}{s - \beta + i} \qquad ,$$

where β is a real constant; $\xi(s)$ maps the half-plane $\operatorname{Im} s \geq 0$ onto a circle Ω_ζ in the ζ-plane which has $\zeta = 0$ on its boundary. Now*

$$\zeta = \xi(s) = \xi(\phi^{-1}(z)) \equiv J(z)$$

is a single-valued function, analytic in the domain Ω_z and continuous on the boundary of Ω_z, which maps Ω_z onto Ω_ζ in such a way that the points $z = 0$ and $\zeta = 0$ coincide. By a theorem of Walsh (see [8], p. 36), given any $\varepsilon > 0$, there is a polynomial $P(z) = c_o + c_1 z + \dots + c_n z^n$

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*By $\phi^{-1}(z)$ we mean the inverse function of $\phi(z)$.

such that

$$|J(z) - P(z)| < \varepsilon$$

on the boundary B_z of Ω_z. Since $J(0) = 0$, we have $|P(0)| < \varepsilon$. Thus, setting $P_1(z) = P(z) - c_o$, so that $P_1(0) = 0$, we have

$$|J(z) - P_1(z)| < 2\varepsilon$$

on B_z. It follows that

(21) $$|\xi(\omega) - P_1(\phi(\omega))| < 2\varepsilon \quad ,$$

since $\phi(s)$ maps the real axis into B_z.

Suppose now that $H(\omega)$ satisfies (20) for $n \geq 1$. Then, using (20) and (21), we have

$$\left| \int_{-\infty}^{\infty} \frac{H(\omega)}{\omega - \beta + i} \, d\omega \right| = \left| \int_{-\infty}^{\infty} H(\omega) \left\{ \xi(\omega) - P_1(\phi(\omega)) \right\} d\omega \right| < 2\varepsilon \int_{-\infty}^{\infty} |H(\omega)| d\omega.$$

Since $H(\omega) \in L^1$ and ε is arbitrary, it follows that

(22) $$\int_{-\infty}^{\infty} \frac{H(\omega)}{\omega - \beta + i} \, d\omega = 0 \quad .$$

Taking the imaginary part of (22), we obtain

$$\int_{-\infty}^{\infty} \frac{H(\omega)}{1 + (\omega - \beta)^2} \, d\omega = 0 \quad ,$$

since $H(\omega)$ is real. Recalling that β is an arbitrary real number, we see that the convolution of $H(\omega)$ and $1/(1 + \omega^2)$ vanishes identically. It follows from the convolution theorem for Fourier transforms that

$h(\tau)\exp(-|\tau|)$ is a null function, where $h(\tau)$ is the Fourier transform of $H(\omega)$. But since $\exp(-|\tau|)$ does not vanish for finite τ, $h(\tau)$ itself and finally $H(\omega)$ is a null function, i.e. aliasing is impossible. (Note that because of the special nature of the mapping $J(z)$, we have not used (20) for the value $n = 0$, so that knowledge of the number $c_h(0)$ is superfluous.) This completes the proof of Theorem 1.

Next we prove a conditional converse of Theorem 1.

Theorem 2. If the characteristic function $\phi(s)$ takes the same value at two different points of the open upper half-plane, then aliasing occurs with additive random sampling.

Proof. We have $\phi(s_1) = \phi(s_2)$, where $s_1 \neq s_2$ and $\operatorname{Im} s_1 > 0$, $\operatorname{Im} s_2 > 0$. It is asserted that there exists a non-null real even function in $L^1 \cap L^2$ such that (20) is valid for $n \geq 0$. Equivalently,* if $h(\tau)$ is the Fourier transform of $H(\omega)$ and $\mathcal{A}$ is the class of Fourier transforms of L^1 functions, it is asserted that there exists a non-null real even function $h(\tau)$ in $\mathcal{A} \cap L^2$, with $h(0) = \int_{-\infty}^{\infty} H(\omega)d\omega = 0$, such that

$$(23) \qquad \int_{-\infty}^{\infty} h(\tau)p_n(\tau)d\tau = \int_{0}^{\infty} h(\tau)p_n(\tau)d\tau = 0 \quad , \quad n \geq 1 \quad ,$$

where the $p_n(\tau)$ are given by (16). We shall show that the assumption that there are no such $H(\omega)$ and $h(\tau)$ leads to a contradiction, thereby proving the theorem.

Suppose there is no $h(\tau)$ with the specified properties which

*Recall that Fourier transformation is a unitary transformation of L^2 onto itself (see [4], p. 70).

satisfies (23). Then since the class $\mathcal{H}$ of such $h(\tau)$ is dense * in the real function space $L^2(0,\infty)$, it follows that there is no non-null function in $L^2(0,\infty)$ for which (23) is satisfied, and consequently that any function in $L^2(0,\infty)$ can be approximated in L^2-norm by linear combinations of the $p_n(\tau)$, $n \geq 1$ (see [4], p. 37). Equivalently (see footnote, p. 25), the Fourier transform of any function in $L^2(0,\infty)$ can be approximated in L^2-norm by linear combinations of the powers $\phi^n(\omega)$, $n \geq 1$. We choose this function in $L^2(0,\infty)$ to be $\exp(-\tau)$, with Fourier transform $1/(1-i\omega)$; this choice is motivated by the fact that $1/(1-is)$ is a schlicht function. Then we have

$$(24) \qquad \lim_{n\to\infty} \int_{-\infty}^{\infty} \left| P_n(\phi(\omega)) - \frac{1}{1-i\omega} \right|^2 d\omega = 0 \quad ,$$

where

$$P_n(x) = a_{n1}x + a_{n2}x^2 + \dots + a_{nn}x^n$$

is a suitable polynomial with real coefficients.

Now form the integral

$$(25) \qquad \frac{1}{2\pi i} \int_{\Gamma} \frac{1}{s-s_0} \left\{ P_n(\phi(s)) - \frac{1}{1-is} \right\} ds \quad ,$$

---

*To see this, note that the class of step-functions is dense in $L^2(0,\infty)$ (see [4], p. 24) and that each step can be approximated by a ".trapezoidal function" ; in particular, a step at the origin can be approximated by a trapezoidal function vanishing at the origin. Since every trapezoidal function is in $\mathcal{A} \cap L^2$ (see [4], p. 89), a subset of $\mathcal{H}$ is dense in $L^2(0,\infty)$. (Here, and below, we depart from our usual notation and use $L^2(0,\infty)$ to denote the space of real functions $h(\tau)$ for which $\int_0^\infty h^2(\tau)d\tau < \infty$.)

where Im $s_o > 0$ and Γ is a contour like the one used in Theorem 1, i.e. a large semicircle Γ_1 erected on the segment $(-R,R)$ as diameter. By Cauchy's theorem, the integral (25) is just the expression in curly brackets evaluated at $s = s_o$. Furthermore, it is easy to see that the contribution to (25) coming from Γ_1, the semi-circular part of the contour Γ, goes to zero as $R \to \infty$.* Combining these facts and using the Schwarz inequality, we have

$$\left| P_n(\phi(s_o)) - \frac{1}{1 - is_o} \right| = \left| \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{1}{\omega - s_o} \left\{ P_n(\phi(\omega)) - \frac{1}{1 - i\omega} \right\} d\omega \right|$$

$$(26) \quad \leq \frac{1}{2\pi} \left\{ \int_{-\infty}^{\infty} \frac{d\omega}{|\omega - s_o|^2} \right\}^{\frac{1}{2}} \left\{ \int_{-\infty}^{\infty} \left| P_n(\phi(\omega)) - \frac{1}{1 - i\omega} \right|^2 d\omega \right\}^{\frac{1}{2}}$$

$$= k \left\{ \int_{-\infty}^{\infty} \left| P_n(\phi(\omega)) - \frac{1}{1 - i\omega} \right|^2 d\omega \right\}^{\frac{1}{2}} ,$$

where k can be chosen so that the inequality holds uniformly for s_o in any closed bounded subset of the open half-plane Im$s > 0$. Choose this subset to contain the points s_1 and s_2 for which $\phi(s_1) = \phi(s_2)$. Then by (24), if n is large enough, we have both

$$\left| P_n(\phi(s_1)) - \frac{1}{1 - is_1} \right| < k\,\varepsilon$$

and

$$\left| P_n(\phi(s_2)) - \frac{1}{1 - is_2} \right| < k\,\varepsilon \quad .$$

whence

$$\left| \frac{1}{1 - is_1} - \frac{1}{1 - is_2} \right| < 2\,k\varepsilon \quad ,$$

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* As already shown, as $s \to \infty$, $\phi(s) \to 0$ uniformly in the upper half-plane.

or, since ε is arbitrary

$$(27) \qquad \frac{1}{1-is_1} = \frac{1}{1-is_2} \quad .$$

However, (27) is impossible for $s_1 \neq s_2$. This contradiction establishes the existence of a real even $H(\omega)$ in $L^1 \cap L^2$ satisfying (20) for $n \geq 0$. The construction of aliases now proceeds in the usual way, i.e. we introduce the functions $H_+(\omega)$ and $H_-(\omega)$ defined by (3) and (4). Then $H_+(\omega)$ and $H_-(\omega)$, and more generally the functions $H_a(\omega)$ defined by (5), are a set of alias power spectra. This completes the proof of Theorem 2.

From an abstract point of view, the question of whether or not aliasing is present with a given method of sampling is just the question of whether or not the mapping from a space whose elements are zero-mean Gaussian random processes to another space whose elements are zero-mean Gaussian random sequences is a one-to-one mapping. Thus, the fact that in the case of additive random sampling, a decisive role is played by the question of whether or not the mapping

$$z = \phi(s) = \int_0^\infty \exp(is\tau)p(\tau)\, d\tau$$

is one-to-one is, at least retrospectively, not too startling.

The method for explicitly reconstructing the correlation function $C(\tau)$ of the underlying sampled process $x(t)$, in a case where aliasing

is absent (as revealed by Theorem 1) is straightforward. First we note that the functions $\phi^n(\omega)$, $n \geq 1$, and correspondingly the functions $p_n(\tau)$, $n \geq 1$, are linearly independent; for if a linear combination of the $\phi^n(\omega)$ with non-zero coefficients vanishes identically, e.g. $a_{n1}\phi(\omega)+a_{n2}\phi^2(\omega)+\ldots+a_{nn}\phi^n(\omega) \equiv 0$ then, since $\phi(\omega)$ is continuous, factoring this polynomial leads to the contradictory conclusion that $\phi(\omega) \equiv$ const. Next, we apply the orthogonalization procedure to the functions $p_n(\tau)$, $n \geq 1$, obtaining the orthonormal set $q_n(\tau)$, $n \geq 1$, which obey the relations

$$\int_0^\infty q_m(\tau)q_n(\tau)d\tau = \delta_{mn} \quad .$$

We then express the $q_n(\tau)$ in the usual fashion (see e.g. [9], p.22) in terms of linear combinations of the $p_n(\tau)$, i.e.

$$q_n(\tau) = b_{n1}p_1(\tau) + b_{n2}p_2(\tau) + \ldots + b_{nn}p_n(\tau) , \quad n \geq 1 \quad . \tag{28}$$

Multiplying (28) by $C(\tau)$ and integrating from 0 to ∞, we obtain the formula

$$\beta_n = \int_0^\infty q_n(\tau)C(\tau)d\tau = b_{n1}c_h(1) + b_{n2}c_h(2)+ \ldots + b_{nn}c_h(n), \; n \geq 1, \tag{29}$$

relating the generalized Fourier coefficients β_n to the correlation sequence $c_h(n)$. Finally, in terms of the β_n and the $q_n(\tau)$, we have

$$C(\tau) = \sum_{n=1}^{\infty} \beta_n q_n(\tau) , \qquad \tau \geq 0 \quad , \tag{30}$$

where the expansion is valid in the sense of mean convergence (i.e.

convergence in the L^2-norm).* Equivalently, we can develop an expression analogous to (30) for $F(\omega)$, the power spectrum of $x(t)$. Recalling that $C(-\tau) = C(\tau)$ and taking the Fourier transform of (30), we have

$$F(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} C(\tau)\exp(-i\omega\tau)d\tau = \frac{1}{2\pi}\sum_{n=1}^{\infty}\int_{0}^{\infty} C(\tau)\left\{\exp(i\omega\tau)+\exp(-i\omega\tau)\right\}d\tau$$

(31)

$$= \frac{1}{2\pi}\sum_{n=1}^{\infty}\beta_n\left\{\Psi_n(\omega) + \Psi_n(-\omega)\right\} \qquad ,$$

where

$$(32) \quad \Psi_n(\omega) = \int_0^{\infty} q(\tau)\exp(i\omega\tau)d\tau = b_{n1}\phi(\omega) + b_{n2}\phi^2(\omega)+\ldots+b_{nn}\phi^n(\omega), \; n > 1$$

and the coefficients β_n and b_{nm} are the same as in (28) and (29). Since $\phi(-\omega) = \overline{\phi(\omega)}$, we have $\Psi_n(-\omega) = \overline{\Psi_n(\omega)}$, so that the even function defined by the last term of (31) is real, as required. Again the expansion (31) is valid in the sense of mean convergence. (Actually, the validity of (30) and (31) in the sense of mean convergence requires the completeness of the functions $p_n(\tau)$, $n \geq 1$, in $L^2(0,\infty)$. This does not quite follow from our proof of Theorem 1, since there we showed that (20) implies that $H(\omega)$ is a null-function with the aid of the assymption that $H(\omega)$ is in L^1 as well as in L^2. By a more involved argument, one can establish the required completeness of the $p_n(\tau)$ in $L^2(0,\infty)$, i.e. one can drop the hypothesis $H(\omega)\in L^1$. (See [10]).

*Since $C(\tau)$ is continuous by assumption and in practice will generally have further regularity properties, we can expect that the expansion (30) will usually be convergent, or at least summable by arithmetic means. Then, since the number $c_h(0)$ has not been used in forming (30), we can check (30) by calculating $C(0)$ and seeing whether $c_h(0) = C(0)$ holds. We can also expect that (31) will usually be convergent or summable.

5. Alias-free sampling methods.

Using the theory of the preceding section, we now exhibit a number of examples of additive random sampling methods, some of which are alias-free and some of which are not. Consider first "Poisson sampling" corresponding to the choice

$$(33) \qquad \begin{aligned} p(\tau) &= \rho \exp(-\rho\tau) \,, & \tau \geq 0 \,, \\ p(\tau) &= 0 \,, & \tau < 0 \,, \end{aligned}$$

which by (16) leads to

$$(34) \qquad \begin{aligned} p_n(\tau) &= \frac{\rho^n \tau^{n-1}}{(n-1)} \exp(-\rho\tau) \,, & \tau \geq 0 \,, \\ p_n(\tau) &= 0 \,, & \tau < 0 \,, \end{aligned}$$

for $n \geq 1$; with the choice (33) the sampling times are the occurrence times of the events in a Poisson process with average rate ρ. Since the corresponding characteristic function

$$\phi(s) = \frac{\rho}{\rho - is}$$

takes no value more than once on the real axis (or anywhere in the complex plane), it follows by Theorem 1 that Poisson sampling is alias-free.*

Reconstruction of the correlation function $C(\tau)$ of the sampled process $x(t)$ from the correlation sequence $c_h(n)$ is particularly

*That Poisson sampling is alias-free also follows from known proofs of the completeness of the Laguerre functions without recourse to Theorem 1.

simple in this case, as might be expected from the fact that the familiar Laguerre functions are obtained by orthogonalizing the functions $\exp(-\tau/2)$, $\tau \exp(-\tau/2)$, $\tau^2 \exp(-\tau/2), \ldots$, which differ only trivially from (34). First we recall the definition of the Laguerre polynomials (see [9], p. 97)

$$(35) \qquad L_n(\tau) = \frac{e^{\tau}}{n!} \frac{d^n}{d\tau^n} (\tau^n e^{-\tau}) = \sum_{k=0}^{n} (-1)^k \binom{n}{k} \frac{\tau^k}{k!} \quad , \quad n \geq 0 \quad .$$

The $L_n(\tau)$ are orthonormal over $(0,\infty)$ with weight $\exp(-\tau)$, i.e.

$$(36) \qquad \int_0^\infty L_m(\tau) L_n(\tau) \exp(-\tau) d\tau = \delta_{mn} \quad .$$

From (36) we obtain by a change of variables

$$2\rho \int_0^\infty L_m(2\rho\tau) L_n(2\rho\tau) \exp(-2\rho\tau) d\tau = \delta_{mn}$$

so that the functions

$$(37) \qquad q_n(\tau) = \sqrt{2\rho}\, L_{n-1}(2\rho\tau) \exp(-\rho\tau) \quad , \quad n \geq 1 \quad ,$$

are orthonormal over $(0,\infty)$. Then, using (34) and (35), we express the $q_n(\tau)$ in terms of the $p_n(\tau)$, obtaining

$$(38) \qquad q_n(\tau) = \sqrt{2/\rho} \sum_{k=0}^{n-1} (-1)^k \binom{n-1}{k} 2^k p_{k+1}(\tau), \quad n \geq 1.$$

Eq. (38) is the explicit form of (28) in the case of Poisson sampling. Therefore, in this case (30) becomes

$$(39) \qquad C(\tau) = \sum_{n=1}^{\infty} \beta_n q_n(\tau) \quad , \quad \tau \geq 0 \quad ,$$

where (see (29))

$$(40)\qquad \beta_n = \sqrt{2/\rho}\ \sum_{k=0}^{n-1} (-1)^k \binom{n-1}{k} 2^k c_h(k+1), \qquad n \geq 1 \quad .$$

Eqs. (38), (39) and (40) give $C(\tau)$ in terms of the correlation sequence $c_h(n)$. The analogous expression for the power spectrum $F(\omega)$ is (see (31) and (32))

$$F(\omega) = \frac{1}{2\pi} \sum_{n=1}^{\infty} \beta_n \left\{ \Psi_n(\omega) + \Psi_n(-\omega) \right\} \qquad ,$$

where the β_n are given by (40), and

$$\Psi_n(\omega) = \sqrt{2/\rho} \sum_{k=0}^{n-1} (-2)^k \binom{n-1}{k} \phi^{k+1}(\omega)$$

$$= \sqrt{2/\rho} \sum_{k=0}^{n-1} (-2)^k \binom{n-1}{k} \left(\frac{\rho}{\rho - i\omega} \right)^{k+1}$$

$$= - \sqrt{2\rho}\ \frac{(i\omega+\rho)^{n-1}}{(i\omega-\rho)^n} , \qquad n \geq 1 \quad .$$

Next consider <u>alternate Poisson sampling</u>, where we sample only at <u>every other</u> point generated by a Poisson process. In this case, $p(\tau)$ is the function $p_2(\tau)$ of (34), with characteristic function

$$\phi(s) = \frac{\rho^2}{(\rho - is)^2} \qquad .$$

Again, it is easy to see that this function takes no value more than once on the real axis, so that alternate Poisson sampling is also

alias-free. However, remarkably enough, if we sample at <u>every third time</u> generated by a Poisson process, aliasing creeps in. In this case, $p(\tau)$ is the function $p_3(\tau)$ of (34), with characteristic function

$$\phi(s) = \frac{\rho^3}{(\rho - is)^3} \quad .$$

Since this function takes the same value at the points $s_1 = \rho(2\sqrt{3} + i)$ and $s_2 = \rho(-2\sqrt{3} + i)$, Theorem 2 guarantees the existence of aliases is this case.

Let $\mathcal{P}$ be the class of probability densities in L^2 for which

$$p(\tau) = 0 \quad , \qquad\qquad \tau < 0 \quad ,$$

$$\int_0^\infty \tau p(\tau) d\tau < \infty$$

(corresponding to a non-zero rate of sampling), and whose Fourier transforms take no value more than once on the real axis (and hence are schlicht in the closed upper half-plane). Each function of $\mathcal{P}$ characterizes an alias-free method of additive random sampling. There are various ways of finding members of $\mathcal{P}$. For example, taking the real and imaginary parts of the condition

$$\int_0^\infty p(\tau) \exp(i\omega_1 \tau) d\tau = \int_0^\infty p(\tau) \exp(i\omega_2 \tau) d\tau \quad ,$$

we obtain

$$(41) \qquad \int_0^\infty p(\tau)\cos\omega_1\tau\,d\tau = \int_0^\infty p(\tau)\cos\omega_2\tau\,d\tau$$

and

$$(42) \qquad \int_0^\infty p(\tau)\sin\omega_1\tau\,d\tau = \int_0^\infty p(\tau)\sin\omega_2\tau\,d\tau \ .$$

If the sine transform is positive for $\omega > 0$ and if the cosine transform is strictly decreasing, then, because of the different parities of the cosine and sine transforms, (41) and (42) cannot both hold for $\omega_1 \neq \omega_2$. Moreover, if $p(\tau)$ is strictly decreasing, it follows from

$$\int_0^\infty p(\tau)\sin\omega\tau\,d\tau = \int_0^{\pi/\omega}\left[p(\tau) - p(\tau+\tfrac{\pi}{\omega}) + p(\tau+\tfrac{2\pi}{\omega}) - \ldots\right]\sin\omega\tau\,d\tau$$

that its sine transform is positive for $\omega > 0$. Thus, the class $\mathcal{P}_0$ of strictly decreasing $p(\tau)$ with strictly decreasing cosine transforms is a subclass of $\mathcal{P}$. Functions in $\mathcal{P}_0$ can be found by inspecting tables of Fourier transforms. For example, $\rho\exp(-\rho\tau)$, $(2\rho/\pi)\exp(-\rho^2\tau^2/\pi)$ and $(2k\rho/\pi)\,\mathrm{sech}(k\rho\tau)$ are all in $\mathcal{P}_0$; here, as usual, ρ is the average sampling rate, and k is the constant

$$\frac{4}{\pi}\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)^2} \cong 1.17 \qquad .$$

Let $p(n;\tau)$, $n = 1,2,\ldots$, be a family of probability densities in $\mathcal{P}_0$, with

$$\int_0^\infty \tau p(n;\tau)\,d\tau = h_n \quad .$$

Then it is easily seen that any function of the form

$$p(\tau) = \sum_{n=1}^{\infty} a_n p(n;\tau) \quad , \quad 0 \le a_n \le 1 \, , \sum_{n=1}^{\infty} a_n = 1$$

is also in $\mathcal{P}_o$, provided that

$$\int_0^{\infty} \tau p(\tau) d\tau = \sum_{n=1}^{\infty} a_n h_n < \infty \quad .$$

More generally, let $p(\alpha;\tau)$, $\alpha \ge 0$, be a family of functions in $\mathcal{P}_o$, indexed by the continuous parameter α, with

$$\int_0^{\infty} \tau p(\alpha;\tau) d\tau = h(\alpha) \quad .$$

Then any function of the form

$$p(\tau) = \int_0^{\infty} p(\alpha;\tau) d\Phi(\alpha) \quad ,$$

where $\Phi(\alpha)$ is a probability distribution function, is also in $\mathcal{P}_o$, provided that

$$\int_0^{\infty} \tau p(\tau) d\tau = \int_0^{\infty} h(\alpha) \, d\Phi(\alpha) < \infty \quad .$$

From the foregoing, it is clear that alias-free sampling methods exist in abundance.

We conclude by noting that the rectangular distribution with probability density

(43)
$$p(\tau) = \frac{1}{2h} \quad , \quad 0 \le \tau \le 2h \quad ,$$
$$p(\tau) = 0 \quad , \quad \tau < 0 \; , \; \tau > 2h \; ,$$

and characteristic function

$$(44) \qquad w = \phi(s) = \frac{\exp(2ihs) - 1}{2ihs} ,$$

is not alias-free. To see this, we construct the image Ω in the w-plane of the first quadrant of the s-plane under the mapping (44). Ω is shown schematically in the figure; it has infinitely many disjoint components separated from one another by the spiral curve (which passes repeatedly through the origin), and the outermost component is bounded by the interval (0,1), the image of the positive imaginary axis. Now consider two points w_1 and w_2 which, like those indicated, lie in different components of Ω. These points correspond to two points s_1 and s_2 of the half-plane $\mathrm{Im}\, s > 0$, and the straight line segment joining s_1 and s_2 lies entirely in $\mathrm{Im}\, s > 0$. Moreover, since the image of this segment must intersect the spiral at a point w_3, there must be a point on the segment, s_3 say, at which $\phi(s)$ takes the same value w_3 as it does at some real $\omega_0 > 0$. Then, since $w = \phi(s)$ is analytic at s_3 and therefore takes all values sufficiently close to $w_3 = \phi(s_3)$ in a neighborhood of s_3 (see [7], p. 184), and since $\phi(s)$ is continuous at ω_0, we can find some value near w_3 which is taken twice in the open upper half-plane. Hence, by Theorem 2, the "rectangular sampling scheme" with distribution (43) is not alias-free.

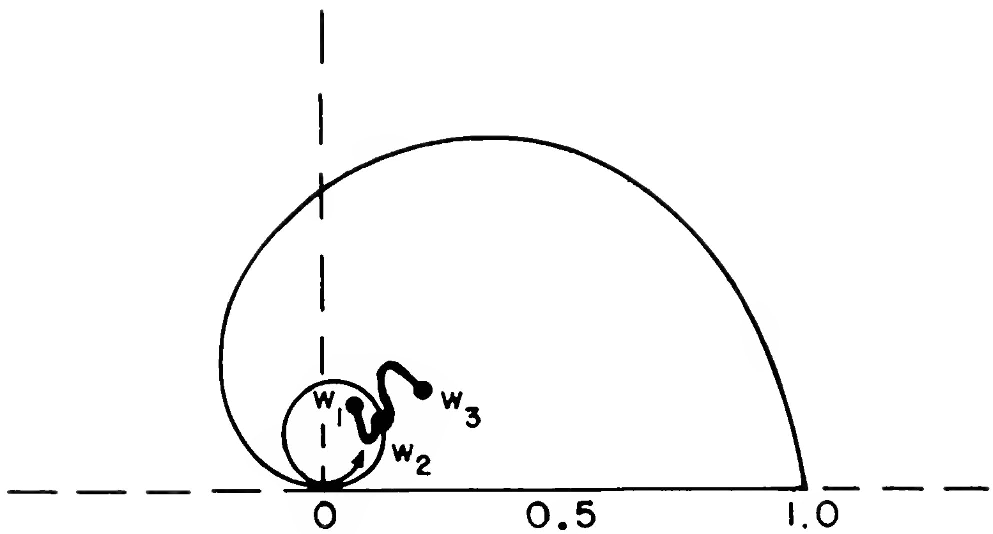

Image in the w-plane of the first quadrant of the s-plane under the mapping (44) with h = 1 (schematic).

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